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# SCALED VISUAL CURVATURE AND VISUAL FRENET FRAME FOR SPACE CURVES

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Abstract. In this paper we define scaled visual curvature and visual Frenet frame that can be visually accepted for discrete space curves. Scaled visual curvature is relatively simple compared to multi-scale visual curvature and easy to control the influence of noise. We adopt scaled minimizing directions of height functions on each neighborhood. Minimizing direction at a point of a curve is a direction that makes the point a local minimum. Minimizing direction can be given by a small noise around the point. To reduce this kind of influence of noise we exmine the direction whether it makes the point minimum in a neighborhood of some size. If this happens we call the direction scaled minimizing direction of $C$ at $p \in C$ in a neighborhood $B_r(p)$.

Normal vector of a space curve is a second derivative of the curve but we characterize the normal vector of a curve by an integration of minimizing directions. Since integration is more robust to noise, we can find more robust definition of discrete normal vector, visual normal vector. On the other hand, the set of minimizing directions span the normal plane in the case of smooth curve. So we can find the tangent vector from minimizing directions. This lead to the definition of visual tangent vector which is orthogonal to the visual normal vector. By the cross product of visual tangent vector and visual normal vector, we can define visual binormal vector and form a Frenet frame.

We examine these concepts to some discrete curve with noise and can see that the scaled visual curvature and visual Frenet frame approximate the original geometric invariants.

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## 1. Introduction

There have been many attempts to find discrete curvature of space curves inherited from smooth curve. Several method was proposed such as curve fitting [6], osculating circle [4], angular defect [2], three point approximation [1].

There are some ways to define discrete Frenet frame for discrete curves. Simple method is to use the forward differences. For example, we can define Frenet frame for a discrete curve(ploygon).[8]

$$C = [\cdots, p_{k-1}, p_k, p_{k+1}, \cdots]$$

$$\begin{aligned} t_k &= \frac{\Delta p_k}{||\Delta p_k||} \\ b_k &= \frac{t_{k-1} \times t_k}{||t_{k-1} \times t_k||} \\ n_k &= b_k \times t_k \end{aligned}$$

where $\Delta p_k$ is the forward difference $\Delta p_k = p_{k+1} - p_k$.(ref. [8]) This can be an approximation to the Frame frame of a smooth curve. In other words, if the points of the polygon $C$ are on the smooth curve $\beta$ and $||\Delta p_k|| \to 0$, then the discrete Frenet frame $\{t_k, n_k, b_k\}$ the polygon $C$ converges to the (original) Frenet frame $\{T, N, B\}$ of $\beta$ in the order 1.

The weakness of this definition is that it assumes an interpolation curve which passes the vertices of the given polygon. But in many applications, the original curve may not be an interpolation of the polygon. For example, if $C$ is a data obtained by a 3-dimensional scan then the point data are not on the original 3D curve but near the original curve. So this simple definition of discrete Frenet frame can not be applied to this kind of discrete curves and we need some other definition of Frenet apperatus which can be applied to the approximation curve data.

Most of these method do not pay much attention to the noise in the discrete curve. The noise can be added to a discrete curve in the process of sampling. So the method to reduce the influence of noise is needed. One of the ways to reduce the influence of noises and errors in the analysis of discrete curve is to redefine the geometric invariants by averaging the behavior of nearby points and nearby data. An attempt was made in [7] by using height functions and multi-scale visual curvature.

In this article we use the height functions defined by directions to define geometric invariants of curves. The difference compared to the other method is that it uses a kind of integration to define differential geometric invariants. So it can have advantages in dealing with noisy data to calculate the geometric invariants such as curvature, torsion and Frenet frame. The purpose of this article is to define a robust discrete curvature and Frenet frame(tangent vector, normal vector and binormal vector) using height functions. In [5] and [7], multi-scale visual curvature was defined to reduce the influence of noises in a complex way. In this paper we introduce a simple way of scaling. We define a scaled minimizing direction and scaled visual curvature that can reduce the influence of noises by the choice of minimizing direction which can make the point minimum in a neighborhood of fixed radius. We also define the visual Frenet frame by using the scaled minimizing directions. And we apply these notions to some discrete curve with noise and we can see that how much the influence of the noise can be reduced by the scaled visual curvature and visual Frenet frame.

## 2. Curvature and Frenet frame of smooth curves

We introduce some concepts on space curves needed to study the discrete version of the geometry of space curves.(ref. [3])

Let $\beta$ be a unit speed smooth curve in $E^3$ with parameter $s$. Tangent vector $T(s)$ of the curve $\beta$ at $\beta(s)$ is defined by the velocity of $\beta$

$$T(s) = \beta'(s)$$

The rate of change of the tangent vector $T'(s)$ have two kinds of geometric significance. If $T'(s) \neq 0$, the direction of $T'(s)$ is a normal direction where the curve bent and we call it the normal vector

$$N(s) = \frac{T'(s)}{||T'(s)||}$$

The norm $||T'(s)||$ measure the amount of rate of change of the direction of the curve and we call it the curvature $\kappa(s)$

$$\kappa(s) = ||T'(s)||$$

At a point $p = \beta(s)$ on $\beta$, the tangent vector $T(s)$ and normal vector $N(s)$ of $\beta$ at $p$ spans the osculating plane $\Pi_o(s)$ of $\beta$ at $p$ and form the Frenet frame $T(s), N(s), B(s)$ with the binormal vector $B(s)$ of $\beta$ at $p$.

$$B(s) = T(s) \times N(s)$$

The plane $\Pi_N(s)$ spanned by $N(s)$ and $B(s)$ is called the normal plane of $\beta$. The following suggests a geometric meaning of these planes.

- Osculating plane $\Pi_o(s)$ is the plane on which the curve $\beta$ almost live near the point $\beta(s)$.
- Normal plane $\Pi_n(s)$ is the plane normal to the curve $\beta$ at $\beta(s)$

Since $T(s), N(s), B(s)$ forms a frame at $\beta(s)$, $\Pi_o(s)$ is normal to $B(s)$ and $\Pi_n(s)$ is normal to $T(s)$.

NOTE 2.1. For convenience we can abbreviate the parameter $s$ or insert $p$ instead of $s$ if it is not confusing.

$$T, T(s), T(p)$$

## 3. Scaled visual curvature

Let $C$ be a space curve. For a direction $\alpha \in S^2$ where $S^2$ is the unit sphere in 3-dimensional Euclidean space, the **height function** $H_\alpha$ on $C$ in the direction $\alpha$ is defined by

$$H_\alpha(s) = \alpha \cdot C(s)$$

For a point $x \in C$, the **set of minimizing directions** of $x$ is defined by

$$A(x) = \{\alpha \in S^2 \mid x \text{ is a strict local minimum point of } H_\alpha\}$$

For small $r > 0$, the **turning angle** of $C$ at $p \in C$ is defined by

$$\theta_r(p) = \frac{1}{2} area\left(\cup_{x \in C \cap B_r(p)} A(x)\right)$$

where $B_r(p) \subset E^3$ is the open ball of radius $r$ centerd at $p$.

The following theorem was proved in [5] to find an approximation of the curvature of space curves.

THEOREM 3.1. (ref. [5]) *For a smooth curve $C$, we have*

$$\kappa(p) = \lim_{r \to 0} \frac{\theta_r(p)}{2r}$$

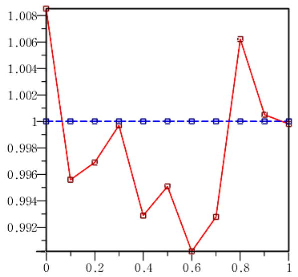
*for every $p \in C$ where $\kappa(p)$ is the curvature of $C$ at $p$.*

In view of this theorem, curvature $\kappa(p)$ can be approximated by

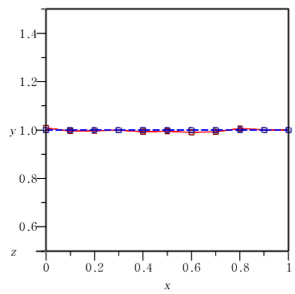
$$\kappa(p) \cong \frac{\theta_r(p)}{2r}$$

for sufficiently small $r$. This leads to the definition of visual curvature of space curves.

For discrete curves there may be noises. So when we consider the minimizing directions, the micrescopic minimizing direction may not be




with noise.png




with noise1-1.png

FIGURE 1. left: close look on the vertical, right: 1-1 ratio

a local minimizing direction due to the noises. For example, Figure 1 is a line and line with noise. (Be sure to checkout the scale. In the left figure, the vertical axis is almost 500 times the horizontal axis. The figure in the right is in the 1-1 scale.) Blue dashed polygon is the original line segment and the red solid polygon is a variation of this line by noise. In the red polygon, for the direction $\vec{u} = (0, 1, 0)$ the fifth point is a local minimum if the neighborhood is consisted of three points. But its is not a local minimum in the neighborhood including five point before and after this point. In this case, three points are not enough as a neighborhood for a discrete curve with noise.

In the computation of discrete geometric invariants such as discrete curvature, too small neighborhood can be much influenced by the noises.

Hence if we want to reduce the influence of noise in the study of discrete curves, we need to calculate geometric objects on a neighborhood of some size.

Discrete analogue of 3-dimensional directions $S^2$ is given by the vertices $V_n$ of the polyhedron $\Lambda_n = (V_n, E_n, F_n)$ constructed through the subdivision of icosahedron inscribed in the unit sphere.(ref. [5]) Using the set of directions $V_n$ which consist of almost uniformly distributed unit vectors in $E^3$, we can define the set of visual minimizing directions.

Definition 3.2. Let $C$ be a discrete space curve, then the **set of scaled minimizing directions** at $x \in C$ in the neighborhood $B_r(x)$ with respect to the polyhedra $\Lambda_n = (V_n, E_n, F_n)$ is defined by

$$A_{n,r}(x) = \{\alpha \in V_n | x \text{ is a strict minimum of } H_\alpha \text{ in } B_r(x)\}$$

for $n, r > 0$.

Note that

$$A_{n,r}(x) \subset V_n \cap A(x)$$

and $A_{n,r}(x)$ can be a proper subset of $V_n \cap A(x)$.

The local union $SM_{n,r}(p)$ at $p \in C$ of scaled minimizing directions is related to many geometric invariants.

$$SM_{n,r}(p) = \cup_{x \in C \cap B_r(p)} A_{n,r}(x)$$

for some small $r > 0$ and large $n > 0$.

With this scaled minimizing directions, we can define the scaled turning angle and scaled visual curvature.

Definition 3.3. For a discrete space curve $C$, **scaled turning angle** $\theta_{n,r}(p)$ and **scaled visual curvature** $\kappa_{n,r}(p)$ at $p \in C$ are defined by the followings:

$$\begin{aligned} \theta_{n,r}(p) &= \frac{2\pi |\ SM_{n,r}(p)|}{|V_n|} \\ \kappa_{n,r}(p) &= \frac{\theta_{n,r}(p)}{2r} \end{aligned}$$

It can be shown that the scaled visual curvature $\kappa_{n,r}(p)$ tends to the curvature $\kappa(p)$ as $n \to \infty,\ r \to 0$. So the scaled visual curvature $\kappa_{n,r}$ can be used as an approximation of the curvature $\kappa$ for sufficiently small $r > 0$ and large $n$. The main benefit of scaled visual curvature is that it can reduce the influence of the noises for an appropriate $r$.

## 4. Characterization of Frenet frame in terms of height functions

In this section we provide a characterization of the Frenet frame of a smooth space curve given by height functions that can be used to define a discrete Frenet frame.

### 4.1. Normal vector

Normal vector of a smooth space curve is a kind of second derivative, but we can characterize the normal vector by an integration by the following lemma proved in [5] for smooth space curves.

LEMMA 4.1. *Let $C$ be a smooth space curve and let $\kappa_x$ be the curvature of $C$ at $x \in C$. If $\kappa_x \neq 0$, then for each $\varphi \in (-\pi/2, \pi/2)$, $x$ is a local minimum of the height function $H_u$ on $C$ defined by the vector $u = \cos\varphi \cdot N_x + \sin\varphi \cdot B_x$ where $N_x,\ B_x$ are the principal normal and binormal vectors of $C$ at $x$, respectively.*

So in the situation of Lemma 4.1, the set of minimizing direction $A(x)$ of $x \in C$ is given by

$$A(x) = \left\{ \cos\varphi \cdot N_x + \sin\varphi \cdot B_x \ \middle|\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \right\}$$

The normal vector of a smooth curve can be expressed by the sum of minimal directions.

THEOREM 4.2. *Let $p$ be a point on a smooth space curve $C$ and the curvature $\kappa_p$ of $C$ at $p$ is not zero, then the (principal) normal vector $N_p$ of $C$ at $p$ is given by the average of minimizing directions of $p$.*

$$N_p = \frac{1}{2} \int_{u \in A(p)} u \tag{4.1}$$

*where $A(p)$ is the set of minimizing directions of $p$.*

*Proof.* By the Lemma 4.1 we have

$$\begin{aligned} \frac{1}{2} \int_{u \in A(p)} u &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(\varphi) d\varphi \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\varphi \cdot N_p + \sin\varphi \cdot B_p)\, d\varphi \\ &= N_p \end{aligned}$$

where $N_p,\ B_p$ are the principal normal and binormal vectors of $C$ at $p$, respectively. □

Note that the normal vector of a curve is defined by a second derivative (for arclength parametrized curves), but the equation (4.1) on normal vectors do not use differentiation but integration. This can lead to a descretization of normal vectors.

### 4.2. Tangent vector

The tangent vector of a smooth curve is normal to the normal plane $\Pi_n(p)$ at $p$ spanned by principal normal vector and binormal vector. For a smooth curve $C$, the set of minimizing directions $A(p)$ at a point $p \in C$ is lying in the normal plane $\Pi_n(p)$. So the tangent vector of a curve can be given by the normal vector to the plane spanned by $A(p)$. But the unit normal vector of a plane is not unique and this depends on the parametrization of the curve.

We can characterize tangent vector of a smooth curve by the minimizing directions as follows.

THEOREM 4.3. *Let $C$ be a smooth space curve defined on some open interval containing $0$ with parameter $t$ and the curvature $\kappa_0$ at $t = 0$ is not zero, then we have*

$$T = sign(\vec{d} \cdot \vec{\sigma})\vec{\sigma}$$

*where $\vec{\sigma}$ is a unit normal vector of the plane spanned by $A(p)$, $\vec{d}$ is a displacement vector $\vec{d} = C(t) - C(0)$ for sufficiently small $t > 0$ and $sign(a) = \frac{a}{|a|}$ for a nonzero real number $a$.*

### 4.3. Frenet frame

Now we can characterize the Frenet frame for a smooth space curve without differentiation and by using height functions and averaging by integration.

THEOREM 4.4. *Let $C$ be a smooth space curve defined on an open interval containing $0$ and $A(p)$ is the set of minimizing directions at $p = C(0)$, then the Frenet frame $T, N, B$ at $p$ is given by the followings:*

$$\begin{aligned} T &= sign(\vec{d} \cdot \vec{\sigma})\vec{\sigma} \\ N &= \frac{1}{2}\int_{u \in A(x)} u \\ B &= T \times N \end{aligned}$$

*where $\vec{\sigma}$ is a unit normal vector to the plane spanned by $A(x)$ and $\vec{d}$ is the displacement vector $\vec{d} = C(t) - C(0)$ for sufficiently small $t > 0$.*

## 5. Visual Frenet frame

### 5.1. Visual normal vector

The first vector of Frenet frame we can define is the normal vector of a curve. In the expression of visual curvature using height function, we measure the amount of minimizing directions near a point.

To define a discrete geometric invariants such as discrete normal vector which can be applied to discrete or digital curves, we need an averaging near the point because of the possible errors. Theorem 4.4 suggests the following discrete version of principal normal vector of a space curve.

The normalized local average of the minimizing directions may be an approximation of the principal normal vector at a point.

Definition 5.1. Let $C$ be a space curve, the **visual normal vector** $N_{n,r}(p)$ of $C$ at $p \in C$ is defined by

$$N_{n,r}(p) = \frac{\sum_{u \in\ SM_{n,r}(p)} u}{\|\sum_{u \in SM_{n,r}(p)} u\|}$$

for some small $r > 0$ and large $n > 0$.

Note that we approximate the integration by a sum in the equation (4.1).

### 5.2. Visual tangent vector and Frenet frame

For a smooth curve $C$, the set of minimizing directions $A(x)$ at a point $x \in C$ spans the normal plane of $C$ at $x$ which is orthogonal to the tangent vector of $C$ at $x$. So the tangent vector can be characterized as the normal vector of the plane spanned by the set of minimizing directions $A(x)$. Threrfore we can think about the average of the cross products $u \times v$ for $u, v \in A(x)$ as a normal vector. But this is zero because of the skew symmetry of the cross product.

We need a suitable criterion which have to be choson for the well oriented normal vector between $u \times v$ and $v \times u$.

Definition 5.2. For a smooth curve $C$, the **direction** $\vec{d}$ of $C$ at a point $C(t)$ is given by the displacement vector

$$\vec{d} = C(t + h) - C(t)$$

for small $h > 0$.

For a discrete space curve $C$ given by a sequence of points $\{p_k\}$ and $p_k \in C$, the **direction** of $C$ at a point $p_k \in C$ is given by the average

$$\vec{d} = \sum_{i=1}^{\nu} (p_{k+i} - p_k)$$

for some $\nu > 0$

For a non-smooth curve, discrete curve or digital curve, the minimizing directions may not lie in a plane. So the plausible candidate for the approximation of tangent vector is the average of the perpendicular vectors of every pairs of vectors in the set of minimizing directions.

Let us use the sign function on real numbers and define some function and sets.

$$sign(a) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases}$$

For $u, v \in SM_{n,r}(p)$, we define the orientation of $(u, v)$ with respect to $\vec{d}$ by

$$\sigma(u, v) = sign(\vec{d} \cdot u \times v)$$

According to the Theorem 4.3, we can define a discrete tangent vector of a discrete space curve.

Definition 5.3. For a discrete space curve $C$, the **visual tangent vector** $T_{n,r}$ of $C$ at $p \in C$ is defined by

$$T_{n,r}(p) = \frac{\sum_{v \in SM_{n,r}(p)} \sigma(N, v) N \times v}{\| \sum_{v \in SM_{n,r}(p)} \sigma(N, v) N \times v \|}$$

for some small $r > 0$ and large $n > 0$.

Obviously visual tangent vector is orthogonal to the visual normal vector.

$$T_{n,r}(p) \perp N_{n,r}(p)$$

Remark 5.4. The first candidate of a normal plane of a discrete space curve is the average of the planes spaned by every pair of minimal directions $u, v \in SM_{n,r}(p)$.

$$Span\{u, v\} = \{au + bv | a, b \in \mathbb{R}\}$$

But we can not gaurantee that this plane contains the visual normal vector $N_{n,r}(p)$. So we choose the plane containing $N_{n,r}(p)$ as a normal plane which is almost normal to the curve.

It is natural to define **visual binormal vector** $B_{n,r}$ by

$$B_{n,r}(p) = T_{n,r}(p) \times N_{n,r}(p)$$

for some small $r > 0$ and large $n > 0$.

Now we have **visual Frenet frame**.

Definition 5.5. Let $C$ be a space curve, the **visual Frenet frame** $T_{n,r}(p),\ N_{n,r}(p),\ B_{n,r}(p)$ of $C$ at $p \in C$ is defined by

$$\begin{aligned}
N_{n,r}(p) &= \frac{\sum_{u\in\ SM_{n,r}(p)} u}{\|\sum_{u\in SM_{n,r}(p)} u\|} \\
T_{n,r}(p) &= \frac{\sum_{v\in SM_{n,r}(p)} \sigma(N_{n,r}(p), v) N_{n,r}(p) \times v}{\| \sum_{v\in SM_{n,r}(p)} \sigma(N_{n,r}(p), v) N_{n,r}(p) \times v\|} \\
B_{n,r}(p) &= T_{n,r}(p) \times N_{n,r}(p)
\end{aligned}$$

Obviously $T_{n,r}(p),\ N_{n,r}(p),\ B_{n,r}(p)$ are orthonormal and forms a frame at $p \in C$.

## 6. Implementations

The number of the set of vertex $V_n$ of the polyhedron $\Lambda_n$ is as follows:

$$|V_1| = 12, |V_2| = 42, |V_3| = 162, |V_4| = 642, |V_5| = 2562, \cdots$$

We can try to use $V_4$ to compute the visual Frenet apperatus.

Remark 6.1. It is not difficult to compute the number of the vertices, edges, faces of the polyhedron $\Lambda_n = (V_n, E_n, F_n)$ according to the algorithm of the subdivision.

$$\begin{aligned}
|V_n| &= 10 \cdot 4^{n-1} + 2 \\
|E_n| &= 30 \cdot 4^{n-1} \\
|F_n| &= 20 \cdot 4^{n-1}
\end{aligned}$$

Let $C : [-\pi, 2\pi] \to R^3$ be a space curve defined by

$$C(t) = (2\cos t, \sin t, \sin 2t)$$

and Figure 2 is the graph of $C$

For the comparison of cuavature and scaled visual curvature, Frenet frame and visual Frenet frame, we discretize in two ways. Discrete curve $dc$ consists of 160 equally spaced sampling point on the curve $C$.

$$dc[k] = C\left(\frac{2\pi}{160}k\right), \quad k = 0, \cdots, 160$$

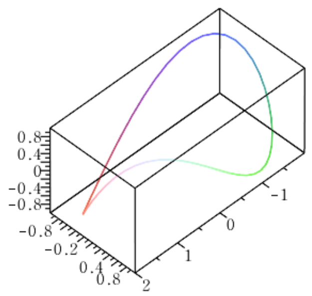



FIGURE 2. graph of $C$

Discrete curve $dcN$ is generated from $dc$ with random noise.

$$dcN[k] = (dc[k]_x + roll(), dc[k]_y + roll(), dc[k]_z + roll())$$

where $roll()$ is a random number function with range $[-0.01, 0.01]$. The ratio of the noise relative to the length of the segment $\overline{dc[k-1]dc[k]}$ is

$$\frac{||dc[k] - dcN[k]||}{||dc[k] - dc[k-1]||} : 0.1 \sim 0.2$$

So the noise are more than 10% the length of the segment and it is not so small.

### 6.1. Comparison of scaled visual curvature

Figure 4 is the graph of the curvature $\kappa$ of $C$. $\kappa$ has minimum $\kappa(\frac{\pi}{2}) \simeq 0.125$ and maximum $\kappa(\frac{3}{4}\pi) \simeq 1.678$($t = \frac{\pi}{2}$ corresponds to $k = 40$ and $t = \frac{3}{4}\pi$ to $k = 60$). So let's estimate scaled visual curvature on these extreme points.

The radius $r$ of the neighborhood $B_r(p)$ is an important factor to reduce the errors from discretization and noise. So we estimate the scaled visual curvature for various $r$.

Figure 5, 6, 7, 8 are the result for the scaled visual curvature of $dc$ and $dcN$ with respect to the direction set $V_4$ and $V_5$. In each case, we can think that $r = 0.3$ to $r = 0.5$ may be a reasonable choice for computation that is about 10% of the diameter of $C$.

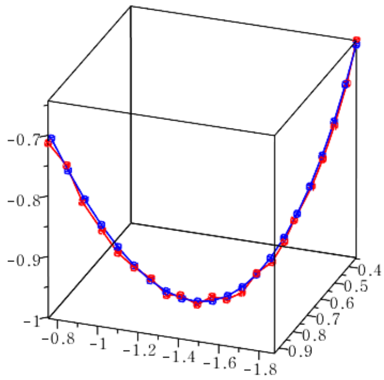



FIGURE 3. the graph of $dc$(blue) and $dcN$(red) from $k = 50$ to $k = 70$

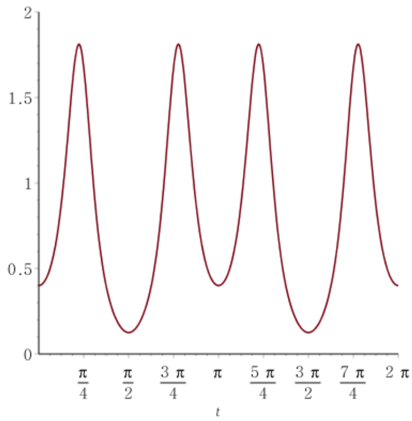



FIGURE 4. the curvature of $C$

| $r$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|
| $dc$ | 0.171 | 0.179 | 0.184 | 0.186 | 0.155 |
| $dcN$ | 1.639 | 0.424 | 0.208 | 0.147 | 0.147 |

FIGURE 5. visual curvature of $dc$ and $dcN$ at $t = \frac{\pi}{2}$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_4$

### 6.2. Comparison of visual Frenet frame

The following tables are the estimation of the differences between Frenet frame of $C$ and visual Frenet frame of discrete curves $dc$ and

| $r$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|
| $dc$ | 0.129 | 0.135 | 0.156 | 0.164 | 0.157 |
| $dcN$ | 1.58 | 0.466 | 0.224 | 0.135 | 0.143 |

FIGURE 6. visual curvature of $dc$ and $dcN$ at $t = \frac{\pi}{2}$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_5$

| $r$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|
| $dc$ | 1.737 | 1.403 | 1.395 | 1.439 | 1.272 |
| $dcN$ | 1.908 | 1.419 | 1.346 | 1.439 | 1.305 |

FIGURE 7. visual curvature of dc and dcN at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_4$

| $r$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|
| $dc$ | 1.711 | 1.435 | 1.413 | 1.420 | 1.275 |
| $dcN$ | 1.950 | 1.447 | 1.343 | 1.405 | 1.306 |

FIGURE 8. visual curvature of dc and dcN at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_5$

| $r$ | $\|\|T_{n,r} - T\|\|$ | $\|\|N_{n,r} - N\|\|$ | $\|\|B_{n,r} - B\|\|$ |
|---|---|---|---|
| 0.2 | 0.0063737209 | 0.0362489706 | 0.0367926590 |
| 0.3 | 0.0207083988 | 0.0292804505 | 0.0339332541 |
| 0.4 | 0.0141485989 | 0.1143342159 | 0.1151618310 |
| 0.5 | 0.0150711665 | 0.1240141067 | 0.1249176469 |
| 0.6 | 0.0275783098 | 0.1501893212 | 0.1502312866 |

FIGURE 9. Errors of visual Frenet frame of $dc$ at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_4$

$dcN$ on the highly curved point $C(\frac{3\pi}{4})$. We can see that the errors are very small. Especially the visual binormal vector is almost the same as the binormal vector. Even for the curve with noise $dcN$, if we use the direction set $V_5$ then the errors are reduced. This is a good sign to compute the torsion from visual binormal vectors.

Figure 13 is a comparison of Frenet frames and visual Frenet frame computed at $t = \frac{\pi}{2}$ for $r = 0.3$. Green arrows are the original Frenet frame and red arrows are the visual Frenet frame. The first figure is for the discrete curve without noise and the second one is for the discrete

| $r$ | $\|\|T_{n,r} - T\|\|$ | $\|\|N_{n,r} - N\|\|$ | $\|\|B_{n,r} - B\|\|$ |
|---|---|---|---|
| 0.2 | 0.0280021748 | 0.0298417961 | 0.0104300432 |
| 0.3 | 0.0427893425 | 0.0431578939 | 0.0105569488 |
| 0.4 | 0.1073525604 | 0.1077334445 | 0.0092225186 |
| 0.5 | 0.1243922458 | 0.1250471759 | 0.0153994104 |
| 0.6 | 0.1386756479 | 0.1386534343 | 0.0193921373 |

FIGURE 10. Errors of visual Frenet frame of $dc$ at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_5$

| $r$ | $\|\|T_{n,r} - T\|\|$ | $\|\|N_{n,r} - N\|\|$ | $\|\|B_{n,r} - B\|\|$ |
|---|---|---|---|
| 0.2 | 0.0383752660 | 0.0815392371 | 0.0838795553 |
| 0.3 | 0.0973243114 | 0.1241313283 | 0.1577155270 |
| 0.4 | 0.0448558156 | 0.1201997691 | 0.1260177116 |
| 0.5 | 0.0312664971 | 0.1277844949 | 0.1308688386 |
| 0.6 | 0.0467746360 | 0.1734211681 | 0.1739159952 |

FIGURE 11. Errors of visual Frenet frame of $dcN$ at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_4$

| $r$ | $\|\|T_{n,r} - T\|\|$ | $\|\|N_{n,r} - N\|\|$ | $\|\|B_{n,r} - B\|\|$ |
|---|---|---|---|
| 0.2 | 0.0721382400 | 0.0358854178 | 0.0628570149 |
| 0.3 | 0.0807831468 | 0.1534088629 | 0.1314660284 |
| 0.4 | 0.1114275744 | 0.1152796563 | 0.0334638656 |
| 0.5 | 0.1327347162 | 0.1343683176 | 0.0256676292 |
| 0.6 | 0.1602286246 | 0.1655283115 | 0.0553901561 |

FIGURE 12. Errors of visual Frenet frame of $dcN$ at $t = \frac{3}{4}\pi$ for $r = 0.2, \cdots, 0.6$ with respect to the direction $V_5$

curve with noise. We can see that the visual Frenet frame of the discrete curve without noise is almost same and the visual Frenet frame of discrete curve with noise is slightly different but not that much.

Figures 14 is the visual Frenet frames at $t = \frac{3}{4}\pi$.

## 7. Conclusion

The visual curvature and multi-scale visual curvature suggested in [7], [5] is intended to design to reflect the human visual perception. But the multi-scale visual curvature is very complicated and not easy to use.

FIGURE 13. visual Frenet frame at $t = \frac{1}{2}\pi$ for $r = 0.3$ with respect to the direction $V_4$. left: $dc$, right: $dcN$

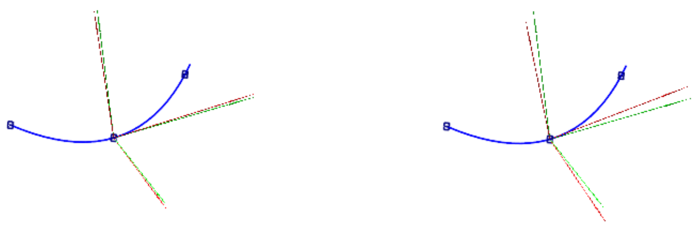


FIGURE 14. visual Frenet frame at $t = \frac{3}{4}\pi$ for $r = 0.3$ with respect to the direction $V_4$. left: $dc$, right: $dcN$

In this article, we suggest a simple way to adopt the scale and define the scaled minimal directions. With this scaled minimal direction scaled visual curvature and visual Frenet frame is defined in a natural way. We can find that the visual Frenet frame is a good approximation to the original Frenet frame even under the noisy situation. Using the visual binormal vector field we may find a multi-level approximation of the torsion of the original curve which is robust under noise.

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